JOURNAL OF APPROXIMATION THEORY 22, 223–232 (1978)

The L_1 Saturation Class of the Kantorovič Operator

VOLKER MAIER*

Abteilung Mathematik, Universität Dortmund, Postfach 500500, D-4600 Dortmund 50, West Germany

Communicated by G. G. Lorentz

Received August 2, 1976

1. INTRODUCTION AND RESULTS

If F is a real-valued function on the interval $I = [0, 1]$, the nth Bernstein polynomial $B_n(F)$ of F is

$$B_n(F, x) = \sum_{k=0}^{n} F(k/n)\, p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

A modification of the Bernstein polynomials due to Kantorovič [4] makes it possible to approximate functions $f \in L_1(I)$ ($L_1(I)$ is the linear space of real-valued Lebesgue integrable functions with the usual L_1 norm) by polynomials, namely by

$$P_n(f, x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt.$$

Let F denote the indefinite integral $\int_0^x f(t)\, dt$. Then

$$\frac{d}{dx} B_{n+1}(F, x) = P_n(f, x) \tag{1}$$

and thus

$$\operatorname{var}_{[0,1]} (B_{n+1}(F, \cdot) - F(\cdot)) = \int_0^1 | P_n(f, x) - f(x)|\, dx.$$

For $f \in L_1(I)$ Lorentz [5] proved in his dissertation that

$$\int_0^1 | P_n(f, x) - f(x)|\, dx \to 0 \ (n \to \infty).$$

* This paper is part of the author's dissertation.

0021-9045/78/0223–0223$02.00/0

He also obtained there the following result where AC(I) denotes the class of real-valued absolutely continuous functions on I.

THEOREM 1. *$F \in \mathrm{AC}(I)$ if and only if*

$$\lim_{n\to\infty} \mathrm{var}_{[0,1]}(B_n(F,\cdot) - F(\cdot)) = 0.$$

The following quantitative version of one part of Lorentz' result is due to Hoeffding [3].

THEOREM 2. *If F is the difference of two convex absolutely continuous functions on I and $J(F') = \int_0^1 x^{1/2}(1-x)^{1/2} \, |df(x)|$ is finite, then*

$$\mathrm{var}_{[0,1]}(B_n(F,\cdot) - F(\cdot)) = O(n^{-1/2}).$$

Hoeffding obtained Theorem 2 as a corollary to the following

THEOREM 3. *If f is a Lebesgue integrable function of bounded variation inside* (0, 1), *then*

$$\int_0^1 |P_n(f,x) - f(x)| \, dx \leqslant (2/e)^{1/2} J(f) \, n^{-1/2},$$

where $J(f) = J(F')$ (see Theorem 2).

Inverse theorems and a "local" version of the saturation are due to Ditzian and May [1].

In this paper we deal with the "global" version of the saturation. We determine the saturation class of the Kantorovič operator and of the Bernstein polynomials in the L_1 norm and in the variation, respectively. Let us denote by BV(I) the class of functions of bounded variation on I. Then our result is

THEOREM 4. *For $f \in L_1(I)$ and $F(x) = \int_0^x f(t)\,dt$ the following two statements are equivalent:*

(i) $\mathrm{var}_{[0,1]}(B_{n+1}(F,\cdot) - F(\cdot)) = \int_0^1 |P_n(f,x) - f(x)| \, dx = O(n^{-1})$,

(ii) *$F \in \mathrm{AC}(I)$ and $F' \doteq f$, $\quad f \in S$,*

$$S := \left\{ f : f(x) \doteq k + \int_\xi^x \frac{h(t)}{t(1-t)} \, dt,\ \xi \in (0,1),\ k \in \mathbb{R} \text{ and } h \in BV(I),\ h(0) = h(1) = 0 \right\}.$$

Moreover, if

(iii) $\mathrm{var}_{[0,1]}(B_{n+1}(F,\cdot) - F(\cdot)) = \int_0^1 |P_n(f,x) - f(x)| \, dx = o(n^{-1})$,

then f is constant a.e.

2. Some Lemmas

The proof of Theorem 4 is based on three lemmas. But first we will give an often used equality. A simple calculation shows that expressed in terms of the B-function

$$\int_0^1 p_{n,k}(x)\,dx = \int_0^1 \binom{n}{k} x^k(1-x)^{n-k}\,dx$$
$$= \binom{n}{k} B(k+1, n-k+1) = \frac{1}{n+1}. \tag{2}$$

Lemma 1. *If $x \in [0, 1]$ then, for $S_k = \sum_{i=1}^{k} 1/i$, $k \in \mathbb{N}$, and $S_0 = 0$, we get*

$$\sum_{k=0}^{n} (S_n - S_k)\, p_{n,k}(x) = \sum_{k=1}^{n} \frac{(1-x)^k}{k}, \qquad n \in \mathbb{N}.$$

Proof. We have

$$S_k = \sum_{i=1}^{k} \frac{1}{i} = \sum_{i=0}^{k-1} \int_0^1 \xi^i\,d\xi = \int_0^1 \frac{1-\xi^k}{1-\xi}\,d\xi,$$

and it follows that

$$S_n - S_k = \int_0^1 \frac{\xi^k - \xi^n}{1-\xi}\,d\xi.$$

Hence

$$\sum_{k=0}^{n} (S_n - S_k)\, p_{nk}(x) = \int_0^1 \frac{((1-x)+x\xi)^n - \xi^n}{1-\xi}\,d\xi$$
$$= \sum_{k=1}^{n} \binom{n}{k}(1-x)^k \int_0^1 \xi^{n-k}(1-\xi)^{k-1}\,d\xi.$$

Applying (2) in a modified form we obtain the result of Lemma 1.

Lemma 2. *If $0 < x \leqslant 1$, then*

$$\int_0^1 |\, P_n(\ln, x) - \ln x \,|\,dx = O(n^{-1}) \qquad (n \to \infty).$$

Proof. We have

$$P_n(\ln, x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{(k/(n+1))}^{(k+1)/(n+1)} \ln t\,dt$$
$$= p_{n,0}(x) \ln\left(e^{-1}\frac{1}{n+1}\right)$$
$$+ \sum_{k=1}^{n} p_{n,k}(x) \ln\left(e^{-1}\left(1+\frac{1}{k}\right)^k \frac{k+1}{n+1}\right).$$

Since

$$\ln x = \ln(1 - (1 - x)) = -\sum_{k=1}^{\infty} \frac{(1-x)^k}{k}, \qquad x \in (0, 1]$$

and by Lemma 1 it is easily seen that

$$\begin{aligned} P_n(\ln, x) - \ln x &= p_{n,0}(x)\Big(\ln\Big(e^{-1}\frac{1}{n+1}\Big) + S_n\Big) \\ &\quad + \sum_{k=1}^{n} p_{n,k}(x)\Big(\ln\Big(e^{-1}\Big(1+\frac{1}{k}\Big)^k \frac{k+1}{n+1}\Big) \\ &\quad + S_n - S_k\Big) + \sum_{k=n+1}^{\infty} \frac{(1-x)^k}{k} \\ &= p_{n,0}(x)\, r_{n,0} + \sum_{k=1}^{n} p_{n,k}(x)\, r_{n,k} + \sum_{k=n+1}^{\infty} \frac{(1-x)^k}{k}, \quad (3) \end{aligned}$$

where

$$r_{n,0} = \ln\Big(e^{-1}\frac{1}{n+1}\Big) + S_n$$

and

$$r_{n,k} = \ln\Big(e^{-1}\Big(1+\frac{1}{k}\Big)^k \frac{k+1}{n+1}\Big) + S_n - S_k, \qquad n, k \in \mathbb{N}.$$

Next we shall estimate $r_{n,0}$ and $r_{n,k}$ with the monotone increasing sequence

$$C_n = \sum_{k=1}^{n} \Big(\frac{1}{k} - \ln\Big(1 + \frac{1}{k}\Big)\Big). \qquad (4)$$

The limit C of this sequence is known as the Eulerian constant (see Gelfond [2, pp. 85–86]).

$$C = \sum_{k=1}^{\infty} \Big(\frac{1}{k} - \ln\Big(1+\frac{1}{k}\Big)\Big) = \sum_{k=1}^{\infty} \Big(\frac{1}{2k^2} - \frac{1}{3k^3} + - \cdots\Big) > 0. \qquad (5)$$

From (4) we get

$$C_n = S_n - \ln(n+1)$$

and thus

$$\ln\frac{k+1}{n+1} + S_n - S_k = C_n - C_k.$$

From (5) it follows for $k = 1, 2, \ldots, n-1$ that

$$\sum_{j=k+1}^{n} \Big(\frac{1}{2j^2} - \frac{1}{3j^3}\Big) \leqslant C_n - C_k \leqslant \sum_{j=k+1}^{n} \frac{1}{2j^2}$$

and by estimating the sums

$$\frac{1}{2k} - \frac{5}{6k^2} + \frac{1}{3(n+1)^2} - \frac{1}{n+1} \leqslant C_n - C_k \leqslant \frac{1}{2k}. \tag{6}$$

On the other hand we have

$$\ln\left(e^{-1}\left(1+\frac{1}{k}\right)^k\right) = \sum_{i=1}^{\infty} (-1)^i \frac{1}{(i+1)\,k^i}.$$

Thus

$$-\frac{1}{2k} \leqslant \ln\left(e^{-1}\left(1+\frac{1}{k}\right)^k\right) \leqslant -\frac{1}{2k} + \frac{1}{3k^2}. \tag{7}$$

Hence by (6) and (7)

$$-\frac{5}{6k^2} + \frac{1}{3(n+1)^2} - \frac{1}{n+1} \leqslant r_{n,k} \leqslant \frac{1}{3k^2},$$

or

$$|\, r_{n,k} \,| \leqslant \frac{5}{6k^2} + \frac{1}{n+1} \qquad (k = 1, 2,\ldots, n-1). \tag{8}$$

Moreover

$$r_{n,0} = -1 + C_n, \qquad r_{n,n} = \ln\left(e^{-1}\left(1+\frac{1}{n}\right)^n\right),$$

$$|\, r_{n,0} \,| \leqslant \ln 2, \qquad |\, r_{n,n} \,| \leqslant 1 - \ln 2.$$

From (3) we now have

$$|\, P_n(\ln, x) - \ln x \,| \leqslant (1-x)^n \ln 2 + \sum_{k=1}^{n-1} \left(\frac{5}{6k^2} + \frac{1}{n+1}\right) p_{n,k}(x)$$

$$+ (1 - \ln 2)\, x^n + \sum_{k=n+1}^{\infty} \frac{(1-x)^k}{k}.$$

Integrating this inequality and applying (2) we get

$$\int_0^1 |\, P_n(\ln, x) - \ln x \,|\, dx \leqslant \frac{\ln 2}{n+1} + \frac{1}{n+1} \sum_{k=1}^{n-1} \frac{5}{6k^2} + \frac{1}{(n+1)^2}$$

$$+ \frac{1 - \ln 2}{n+1} + \sum_{k=n+1}^{\infty} \frac{1}{k^2}$$

$$\leqslant \frac{1}{n+1} + \frac{1}{n+1}\,\frac{5\pi^2}{36} + \frac{1}{(n+1)^2} + \frac{1}{n}.$$

This proves Lemma 2.

LEMMA 3. *If* $f(x) = \ln(1 - x)$, $x \in (0, 1)$, *then*

$$\int_0^1 | P_n(f, x) - f(x)| \, dx = O(n^{-1}).$$

Proof. A short computation shows that for $f_1(x) = f(1 - x)$ we have

$$\int_0^1 | P_n(f_1, x) - f_1(x)| \, dx = \int_0^1 | P_n(f, x) - f(x)| \, dx.$$

Thus Lemma 3 follows from Lemma 2.

3. PROOF OF THEOREM 4

First we show that (i) ⇒ (ii).

We consider the bilinear functional

$$A_n(f, \psi) = 2n \int_I (P_n(f, x) - f(x)) \, \psi(x) \, dx \tag{9}$$

for $f \in L_1(I)$ and $\psi \in C^2(I)$ ($C^2(I)$ the class of functions which are twice continuously differentiable on I).

First we will treat $A_n(f, \psi)$ as a functional in f. Since

$$B_{n+1}(F, 0) = F(0), \qquad B_{n+1}(F, 1) = F(1) \tag{10}$$

it follows from (9) by partial integration that

$$A_n(f, \psi) = -2n \int_I (B_{n+1}(F, x) - F(x)) \, \psi'(x) \, dx. \tag{11}$$

Let $\psi \in C^2(I)$ then we shall determine $A(\cdot, \psi)$, the limit of the sequence $A_n(\cdot, \psi)$ on $L_1(I)$. Applying the theorem of Banach and Steinhaus (see Wloka [9, p. 126]), we require that the functionals $A_n(\cdot, \psi)$ have uniformly bounded norms on $L_1(I)$. The proof of this fact appears in the author's dissertation [6] and in Ditzian and May [1, Lemma 5.3]. The latter lemma was stated for $\psi \in C_0^\infty(0, 1)$ but also holds equally well in this case. If now $F \in C^2(I)$, there is by a theorem of Voronowskaja [7]

$$\lim_{n \to \infty} A_n(f, \psi) = -\int_I x(1 - x) f'(x) \, \psi'(x) \, dx$$

$$= \int_I f(x)(x(1 - x) \, \psi'(x))' \, dx.$$

Since $C^2(I)$ is dense in $L_1(I)$ and the functionals $A_n(\cdot, \psi)$ have uniformly bounded norms on $L_1(I)$, we have for all $f \in L_1(I)$

$$A(f, \psi) = \int_I f(x)(x(1-x)\,\psi'(x))'\,dx. \tag{12}$$

On the other hand, we are able to rewrite (9) by applying (1)

$$A_n(f, \psi) = \int_I \psi(x)\,d(2n(B_{n+1}(F, x) - F(x)).$$

Let $h_n(x) = n(B_{n+1}(F, x) - F(x))$; then by (i), $h_n \in \mathrm{BV}(I)$ and by (10), $|h_n(x)| = |h_n(x) - h_n(0)| \leqslant \mathrm{var}_{[0,1]}\,h_n = O(1)$ uniformly for all $x \in I$. Applying the theorems of Helly and Bray [8, pp. 29 and 31] we can extract from $h_n(x)$ a subsequence $h_{n_p}(x)$ which converges on I to a function $h(x) \in \mathrm{BV}(I)$ and we have for all $\psi \in C(I)$

$$\lim_{n\to\infty} A_{n_p}(f, \psi) = \int_I \psi(x)\,dh(x), \tag{13}$$

where $h(0) = h(1) = 0$.

From (12) and (13)

$$\int_I f(x)(x(1-x)\,\psi'(x))'\,dx = \int_I \psi(x)\,dh(x). \tag{14}$$

To determine the solution f of this inhomogeneous problem we will first solve the homogeneous part

$$\int_I f(x)(x(1-x)\,\psi'(x))'\,dx = 0 \tag{15}$$

or

$$\int_I f(x)(x(1-x)\,\psi''(x) + (1-2x)\,\psi'(x))\,dx = 0. \tag{16}$$

By partial integration of the second term we get

$$\int_I f(x)(1-2x)\,\psi'(x)\,dx = G(x)\,\psi'(x)\Big|_0^1 - \int_I G(x)\,\psi''(x)\,dx,$$

where $G(x) = \int_o^x (1-2t)\,f(t)\,dt$. Because (15) holds for all $\psi \in C^2(I)$ we may choose $\psi(x) = x$ and then by (15), $G(1)$ becomes zero. Hence

$$\int_I (f(x)\,x(1-x) - G(x))\,\psi''(x)\,dx = 0.$$

Thus we have for the absolutely continuous function G

$$G(x) \doteq f(x)\, x(1-x)$$

or

$$G'(x) = f(x)(1-2x) \doteq f'(x)\, x(1-x) + f(x)(1-2x)$$

and from this $f'(x) \doteq 0$. The general solution for the homogeneous problem (15) is then $f(x) \doteq k$ $(k \in \mathbb{R})$.

A short computation shows that for a function $h \in \mathrm{BV}(I)$, where $h(0) = h(1) = 0$,

$$f(x) = \int_\xi^x \frac{h(t)}{t(1-t)}\, dt$$

is a particular solution for the inhomogeneous problem (14). Altogether we have the general solution for (14)

$$f(x) \doteq k + \int_\xi^x \frac{h(t)}{t(1-t)}\, dt \qquad (k \in \mathbb{R}).$$

This concludes the proof.

Now we shall prove (ii) $\Rightarrow$ (i).

We must estimate $\| P_n f - f \|_1$ and may thereby omit constant terms, because

$$P_n(c) = c \qquad (c \in \mathbb{R}). \tag{17}$$

First we will rewrite $f \in S$ with $h \in \mathrm{BV}(I)$ and $h(0) = h(1) = 0$

$$\begin{aligned} f(x) &= \int_\xi^x \frac{h(t)}{t(1-t)}\, dt \\ &= \int_\xi^1 \frac{h(t)}{t}\, dt - \int_x^1 \frac{h(t)}{t}\, dt + \int_0^x \frac{h(t)}{1-t}\, dt - \int_0^\xi \frac{h(t)}{1-t}\, dt. \end{aligned}$$

If $\xi \in (0, 1)$ is fixed, we have (see text preceding (17)) with $h \in \mathrm{BV}(I)$, $h = h_1 - h_2$ where h_1, h_2 are nondecreasing functions on I,

$$\begin{aligned} f(x) = {} & \int_x^1 \frac{h_2(t)}{t}\, dt - \int_x^1 \frac{h_1(t)}{t}\, dt \\ & + \int_0^x \frac{h_1(t)}{1-t}\, dt - \int_0^x \frac{h_2(t)}{1-t}\, dt. \end{aligned} \tag{18}$$

Let us now consider the function

$$g(x) = \int_x^1 \frac{h_1(t)}{t}\, dt - \int_0^x \frac{h_1(t)}{1-t}\, dt, \qquad x \in (0, 1).$$

For h_1 nondecreasing there is such a $c_1 \in \mathbb{R}$ that for $m_1(t) = h_1(t) + c_1$ $(t \in I)$ $m_1(0) = 0$. Obviously m_1 is nondecreasing and $m_1(t) \geqslant 0$ for $t \in I$. Hence

$$g(x) = \int_x^1 \frac{m_1(t)}{t}\, dt - \int_0^x \frac{m_1(t)}{1-t}\, dt + c_1 \ln x + c_1 \ln(1-x).$$

For

$$g_1(x) = \int_x^1 \frac{m_1(t)}{t}\, dt, \qquad x \in (0, 1],$$

there is

$$g_1(x) - g_1(x_0) \leqslant m_1(x_0)(\ln x_0 - \ln x), \qquad x, x_0 \in (0, 1], \tag{19}$$

and for

$$g_2(x) = -\int_0^x \frac{m_1(t)}{1-t}\, dt, \qquad x \in [0, 1),$$

$$g_2(x) - g_2(x_0) \leqslant m_1(x_0)(\ln(1-x) - \ln(1-x_0)), \qquad x, x_0 \in [0, 1). \tag{20}$$

If x_0 is fixed, we get from (19) after operating with P_n and then writing x for x_0

$$P_n(g_1, x) - g_1(x) \leqslant m_1(x)(\ln x - P_n(\ln, x)).$$

Hence

$$\begin{aligned} | P_n(g_1, x) - g_1(x)| \leqslant & -(P_n(g_1, x) - g_1(x)) \\ & + m_1(x)(\ln x - P_n(\ln, x)) \\ & + | m_1(x)(\ln x - P_n(\ln, x))|. \end{aligned}$$

Since $\int_I P_n(g_1, x)\, dx = \int_I g_1(x)\, dx$ and $m_1(0) = 0$ we have from (19)

$$\int_I | P_n(g_1, x) - g_1(x)|\, dx \leqslant 2 \operatorname{var}_{[0,1]} m_1 \int_I | P_n(\ln, x) - \ln x |\, dx \tag{21}$$

and from (20)

$$\int_I | P_n(g_2, x) - g_2(x)|\, dx \leqslant 2 \operatorname{var}_{[0,1]} m_1 \int_I |(P_n(f_1, x) - f_1(x))|\, dx, \tag{22}$$

where $f_1(x) = \ln(1-x)$.

If we now treat the terms of (18) with h_2 in an analogous way we have a function $m_2(t) = h_2(t) + c_2$ $(c_2 \in \mathbb{R})$ with $m_2(0) = 0$ for $t \in I$.

Altogether we get from (18)

$$\begin{aligned} \int_I | P_n(f, x) - f(x)|\, dx \leqslant k_1 \Big(& \int_I | P_n(\ln, x) - \ln x |\, dx \\ & + \int_I | P_n(f_1, x) - f_1(x)|\, dx \Big), \end{aligned}$$

where $f_1(x) = \ln(1 - x)$ and

$$k_1 = 2\,\mathrm{var}_{[0,1]}\, m_1 + 2\,\mathrm{var}_{[0,1]}\, m_2 + |c_1| + |c_2|.$$

Applying Lemma 2 and Lemma 3 the implication (ii) $\Rightarrow$ (i) is shown.

Proof of (iii). We have $\lim_{n\to\infty} n\, \|P_n f - f\|_1 = 0$ and thus for $\psi \in C^2(I)$

$$\lim_{n\to\infty} 2n \int_I (P_n(f, x) - f(x))\, \psi(x)\, dx = 0.$$

From this it follows (see proof of (i) $\Rightarrow$ (ii)) that

$$\int_I f(x)(x(1 - x)\, \psi'(x))'\, dx = 0,$$

the homogeneous problem (15) with its general solution $f(x) \doteq k$ ($k \in \mathbb{R}$). This concludes the proof.

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